The Wave-making Resistance of Ships: a Theoretical and Practical Analysis.

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CONTENTS.

§ 1. *Introduction and Summary.*

The theoretical investigation of the total resistance to the forward motion of a ship is usually simplified by regarding it as the sum of certain independent terms such as the frictional, wave-making, and eddy-making resistances. The experimental study of frictional resistance leads to a formula of the type

$$R_s = f S V^m, \tag{1}$$

where S is the wetted surface, V the speed, f a frictional coefficient, and m an index whose value is about 1·83.

After deducting from the total resistance the frictional part calculated from a suitable formula of this kind, the remainder is called the residuary resistance. Of this the wave-making resistance is the most important part; the present paper is limited to the study of wave-making resistance, and chiefly its variation with the speed of the ship. The hydrodynamical theory as it stands at present may be stated briefly.

Simplify the problem first by having no diverging waves; that is, suppose the motion to be "in two dimensions in space," the crests and troughs being in infinite parallel lines at right angles to the direction of motion. Further, suppose that the motion was started at some remote period and has been maintained uniform. We know that, except very near to the travelling disturbance, the surface motion in the rear consists practically of simple periodic waves of length suitable to the velocity v of the disturbance. Let

a be the amplitude of the waves, and w the weight of unit volume of water; then the mean energy of the wave motion per unit area of the water surface is $\frac{1}{2}wa^2$. Imagine a fixed vertical plane in the rear of the disturbance; the space in front of this plane is gaining energy at the rate $\frac{1}{2}wa^2v$ per unit time. But on account of the fluid motion, energy is supplied through the imaginary fixed plane to the space in front, and it can be shown that the rate of supply is $\frac{1}{2}wa^2u$, where u is the group-velocity corresponding to the wave-velocity v. The nett rate of gain of energy is $\frac{1}{2}wa^2(v-u)$, and this represents the part of the power of the ship which is needed, at uniform velocity, to feed the procession of regular waves in its rear. An equivalent method of stating this argument is to regard the whole procession of regular waves from the beginning of the motion as a simple group; then the rear moves forward with velocity u while the head advances with velocity v, and the whole procession lengthens at the rate $v-u$. If we write $\mathrm{R}v$ for the rate at which energy must be supplied by the ship, we call R the wave-making resistance, and we have

$$\mathrm{R} = \tfrac{1}{2}wa^2(v-u)/v. \tag{2}$$

We notice that R is the wave-making resistance in *uniform* motion; it is only different from zero because u differs from v, that is, because the velocity of propagation depends upon the wave-length.

In deep water, u is $\frac{1}{2}v$, so that R is $\frac{1}{4}wa^2$. In the application of this to a ship at sea, it is assumed that the transverse waves have a certain average uniform breadth and height, and, further, that the diverging waves may be considered separately and as having crests of uniform height inclined at a certain angle to the line of motion; if the amplitude is taken to vary as the square of the velocity, it follows that R varies as v^4. Several formulæ of the type $\mathrm{R} = \mathrm{A}v^4$, or $\mathrm{R} = \mathrm{A}v^4+\mathrm{B}v^6$, have been proposed; although these may be of use practically by embodying the results of sets of experiments, they are not successful from a theoretical point of view. Recently many such cases have been analysed graphically by Prof. Hovgaard;* the general result is that a fair agreement may be made for lower velocities with an average experimental curve neglecting the humps and hollows due to the interference of bow and stern wave systems, but at higher velocities the experimental curve falls away very considerably from the empirical curve.

The method used here consists in considering the ship, in regard to its wave-making properties, as equivalent to a transverse linear pressure distribution travelling uniformly over the surface of the water. Taking a simple form of diffused pressure system and making some necessary

* W. Hovgaard, 'Inst. Nav. Arch. Trans.,' vol. 50, p. 205, 1908.

assumptions, we obtain an expression for the amplitude of the transverse waves thus originated, and for the resistance R, in which the velocity enters in the form e^{-a/v^2}; this function is seen to have the general character of the experimental curves. Adding on a similar term for the waves diverging from bow and stern, and, finally, in the manner of W. Froude, an oscillating factor for the interference of these bow and stern waves, we find a formula for the wave-making resistance of the type

$$\mathrm{R} = \alpha e^{-l/v^2} + \beta\{1 - \gamma\cos(m/v^2)\}\, e^{-n/v^2}.$$

In this expression there are six adjustable constants; we proceed to reduce the number of these after transforming into units which utilise Froude's law of comparison. We use the quantity c, defined as

(speed in knots)/$\sqrt{}$(length of ship in feet),

and we express the resistance in lbs. per ton displacement of the ship. An inspection of experimental curves, and other considerations suggest that the quantities l, m, n may be treated as universal constants; with this assumption, a three-constant formula is obtained, viz.,

$$\mathrm{R} = \alpha e^{-2\cdot 53/9c^2} + \beta\{1 - \gamma\cos(10\cdot 2/c^2)\}\, e^{-2\cdot 53/c^2}, \tag{3}$$

where the constants α, β, γ depend upon the form of the ship.

We then treat (3) as a semi-empirical formula of which the form has been suggested by the preceding theoretical considerations; several experimental model curves are examined, and numerical calculations are given which show that these can be expressed very well by a formula of the above type.

Since the constant α is found to be small compared with β, it is not allowable to press too closely the theoretical interpretation of the first term, especially as the experimental curves include certain small elements in addition to wave-making resistance. If we limit the comparison to values of c from about 0·9 upwards, it is possible to fit the curves with an alternative formula of the type

$$\mathrm{R} = \beta\{1 - \gamma\cos(10\cdot 2/c^2)\}\, e^{-n/c^2},$$

and some examples of this are given.

The effect of finite depth of water is considered, and a modification of the formula is obtained to express this effect as far as possible. Starting from an experimental curve for deep water, curves are drawn, from the formula, for the transverse wave resistance of the same model with different depths; although certain simplifications have to be made, the curves show the character of the effect, and allow an estimate of the stage at which it becomes appreciable.

In the last section the question of other types of pressure distribution is

discussed, and one is given in illustration of the wave-making resistance of an entirely submerged vessel.

§2. *Pressure System travelling over Deep Water.*

It is known that a line pressure-disturbance travelling over the surface of water with uniform velocity v at right angles to its length gives rise to a regular wave-train in its rear of equal wave-velocity.* Take the axis of x in the direction of motion and let the pressure system be symmetrical with respect to the origin and given by $p = f(x)$; suppose that $f(x)$ vanishes for all but small values of x, for which it becomes infinite so that $\int_{-\infty}^{\infty} f(x)\, dx = \mathrm{P}$. The regular part of the surface depression η due to this integral pressure P practically concentrated on a line is given by

$$\eta = \frac{2g\mathrm{P}}{wv^2} \sin \frac{gx}{v^2}. \tag{4}$$

The part of the surface effect which is neglected in this expression consists of a local disturbance symmetrical with respect to the origin and practically confined to its neighbourhood.

If we suppose P constant, the amplitude in the regular wave-train and the consequent drain of energy due to its maintenance diminish with the velocity.

To obtain results in any way comparable with practical conditions it is necessary to suppose the pressure system diffused over a strip which is not infinitely narrow.

An illustration is afforded by taking

$$p = f(x) = \frac{\mathrm{P}}{\pi} \frac{\alpha}{\alpha^2 + x^2}, \tag{5}$$

where α is small compared with the distances at which the regular surface effects are estimated. This type of pressure distribution is shown in fig. 1.

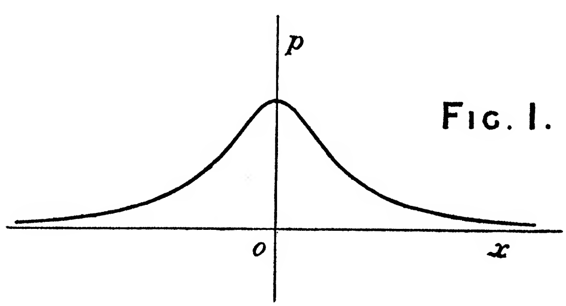


FIG. 1.

* For a discussion of the wave pattern, see Lamb, 'Hydrodynamics,' § 241 *et seq.*; or Havelock, 'Roy. Soc. Proc.,' A, vol. 81, p. 398, 1908.

The effect of thus diffusing the pressure system is expressed by the introduction of a factor $\phi(\kappa)$ into the amplitude of the regular waves, where $2\pi/\kappa$ is the wave-length and

$$\phi(\kappa) = \int_{-\infty}^{\infty} f(\omega)\cos\kappa\omega\, d\omega. \tag{6}$$

Using (5) in (6), we find

$$\phi(\kappa) = \mathrm{P}e^{-\alpha\kappa} = \mathrm{P}e^{-\alpha g/v^2}.$$

Hence the amplitude of the waves is given by

$$a = \frac{2g\mathrm{P}}{wv^2} e^{-\alpha g/v^2}. \tag{7}$$

Further, since $\kappa = v^2/g$, the group velocity $u = d(\kappa v)/d\kappa = \frac{1}{2}v$. Hence the wave-making resistance R is given by

$$\mathrm{R} = \frac{g^2\mathrm{P}^2}{wv^4} e^{-2\alpha g/v^2}. \tag{8}$$

We have to examine the variation of these quantities with the velocity v under the supposition that the pressure system is due to the motion of a body either floating on the surface or wholly immersed in the water. The pressures concerned being the vertical components of the excess or defect due to the motion, it seems possible to assume as a first approximation that P varies as v^2; this is the case in the ordinary hydrodynamical theory of a solid in an infinite perfect fluid, and a similar assumption is also made in the theory of Froude's law of comparison. This being assumed, we find

$$a = \mathrm{A}e^{-\alpha g/v^2}, \qquad \mathrm{R} = \mathrm{B}e^{-2\alpha g/v^2}. \tag{9}$$

We see that both the amplitude and the resistance increase steadily from zero up to limiting values.

If we draw the curve representing this relation between R and v, there is a point of inflection when

$$\frac{d^2\mathrm{R}}{dv^2} = 0, \qquad \text{or} \qquad v^2 = \tfrac{4}{3}g\alpha. \tag{10}$$

Writing v' for this velocity, we see that $d\mathrm{R}/dv$ increases as the velocity rises to v' and then falls off in value as the velocity is further increased.

We can write the relation now in the form

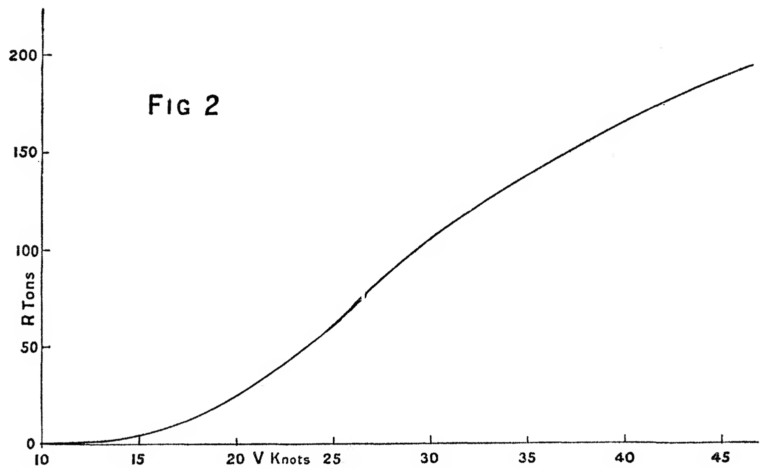
$$\mathrm{R} = \mathrm{B}e^{-\frac{3}{2}(v'/v)^2}. \tag{11}$$

The character of this relation is shown by the curve in fig. 2, which represents the case

$$\mathrm{R} = 315e^{-\frac{3}{2}(26/\mathrm{V})^2}, \tag{12}$$

R being in tons, and V in knots.

The values of the constants in (12) have been chosen for comparison with an experimental curve of residuary resistance given by R. E. Froude ;* it was obtained from model experiments and by means of the law of



corresponding speeds and dimensions the results were given for a ship (model A) of 4090 tons displacement and 400 feet length. The actual curve is given in fig. 4 and is discussed more fully later; we neglect for the present the undulations which are known to be due to the interference of the bow and stern wave systems, and we consider a fairly drawn mean experimental curve denoted by R′. Table I shows a comparison of the values of R′ with those of R calculated from the formula (12).

Table I.

V.	R.	R′.
10	0·02	1·8
14	2	4
18	14	16
22	38	39·5
26	70	70
30	106	107
34	132	130
38	157	156
42	176	175
46	195	192

From this comparison we see that the point of inflection given by V′ orresponds to the point at which the slope of the mean experimental curve

* R. E. Froude, 'Inst. Nav. Arch. Trans.,' vol. 22, p. 220, 1881.

begins to fall off. This effect is general in residuary resistance curves; we see that it is really an interference effect, the character of the curve being due to the mutual interference of the wave-making elements of the pressure system. Superposed on the mean curve we have a further interference effect due to the combination of two systems, the bow and stern systems.

From Table I we infer that the mean curve agrees well with the calculated values R from about 18 knots upwards, but at the lower speeds the values of R are much too small; this suggests the addition of a term to represent the effect of the diverging waves.

§ 3. *Diverging Wave System.*

In the example considered above, the calculated values of R are much too small at the lower velocities. This might have been expected; for we obtained (12) by the consideration of line-waves on the surface, that is waves with crests of uniform height along parallel infinite lines. But the model experiments correspond more to a point disturbance travelling over the surface, with the formation of diverging waves as well as transverse waves. In fact, W. Froude* infers from his experimental curves that the residuary resistance at the lower velocities is chiefly due to the diverging wave system, on account of the absence of undulations; for the latter signify interference of the transverse systems initiated by the bow and stern, and these become very important at the higher velocities.

We have to add to (12) a term representing the diverging waves; the comparison in Table I suggests for this a term of the same type, $e^{-\frac{3}{2}(V''/V)^2}$, with V'' much smaller than the corresponding velocity V' for the transverse waves. With the data at our disposal we might then determine the various constants so as to obtain the closest fit possible; however, we can make the process appear less artificial by the following considerations. We know that the wave pattern produced by a travelling point source consists of a system of transverse waves and a system of diverging waves, the whole pattern being contained with two radial lines making angles of about 19° 28′ with the direction of motion; a fuller investigation of the effects produced by a diffused source must be left over at present. In applying energy considerations as in the previous sections, the usual method is to suppose that the transverse waves form on the average a regular wave-train of uniform amplitude and uniform breadth; using the same approximation for the diverging waves we suppose that these form on the average a regular wave-train on each side, with the crests inclined at some angle θ to the direction

* W. Froude, 'Inst. Nav. Arch. Trans.,' vol. 18, p. 86, 1877.

of motion of the disturbance. Then the velocity of the diverging wave-trains normally to their crests is $V \sin\theta$. Now the same features of the ship are responsible for the character of both transverse and diverging waves; then if V' is the velocity at which there is a point of inflection in the resistance curve for the transverse waves, the suggestion is that $V' \sin\theta$ is the corresponding velocity for the diverging waves. Taking as a first approximation the angle given above, viz., 19° 28′ or $\sin^{-1}\frac{1}{3}$, we test now a formula of the type

$$R = Ae^{-\frac{3}{2}(V'/3V)^2} + Be^{-\frac{3}{2}(V'/V)^2}. \tag{13}$$

For the particular example already used (Froude, Ship A) we take V' equal to 26 knots, and determine A, B from two values of V. We obtain thus

$$R = 4{\cdot}5e^{-\frac{3}{2}(26/3V)^2} + 297e^{-\frac{3}{2}(26/V)^2}. \tag{14}$$

With this formula we find as good an agreement as before at the higher velocities, and we have now at lower velocities the comparison in Table II:—

Table II.

V.	R.	R′.
10	1·6	1·8
14	4·1	4
18	16·5	16
22	40	39·5

In calculating from (14) we find that the two terms both increase continually; at low velocities the second term is practically negligible, then at about 15 knots the two terms are of equal value, and after that the transverse wave term becomes all important.

It must be remembered that the experimental curve was obtained from tank experiments, and it is possible that the width of the tank may have an effect on the relative values of the transverse and diverging waves. It would be of interest if experiments were possible with the same model in tanks of different widths; if the methods used in obtaining (14) form a legitimate approximation, the effect might be shown in the relative proportions of the two terms—provided always that one can make a suitable deduction first for the frictional resistance, and can then separate out the relatively small effects of the diverging waves, the eddy-making and other similar elements.

§ 4. *Interference of Bow and Stern Wave-trains.*

The cause of the undulations in the resistance curves was shown by W. Froude to be interference of the wave system produced by the bow (or entrance) with that arising at the stern (or run). His experiments on the effect of introducing a parallel middle body between entrance and run confirmed his theory, which may be stated briefly. Let the wave-making features of the bow produce transverse waves which would have at a breadth b an amplitude a; owing to the spreading out of the transverse waves they will be equivalent to simple waves at the stern of smaller amplitude ka, at the same breadth b. Let a' be the amplitude there of the waves produced by the stern. Then in the rear of the ship we suppose there are simple waves of amplitude ka superposed upon others of equal wave-length of amplitude a'. At certain velocities the crests of the two systems coincide in position, giving rise to a hump on the resistance curve; and at intermediate velocities there are hollows on the curve owing to the crests of one system coinciding with the troughs of the other.

In developing a form for the resistance, subsequent writers have generally taken R proportional to an expression of the form $a^2+a'^2+2kaa' \cos(mg\mathrm{L}/v^2)$, where L is the length of the ship. This means that the bow is supposed to initiate a system of waves with a first crest at a short distance behind the bow, and that similarly the stern waves have their first crest shortly after the stern; the length mL is the distance between these two crests, and is called the wave-making length of the ship. The determination of a value for m appears to be doubtful, but from interference effects it is said to vary for different ships between the values 1 and 1·2.

It has seemed desirable here to follow more closely the point of view in W. Froude's original paper already quoted.* We regard the entrance of the ship as forming transverse waves with their first crest shortly aft of the bow, and the run of the ship as forming waves with their first trough in the vicinity of the middle of the run. It is suggested that this distance between first crest and first trough, in practice found to be about 0·9L, should be taken as the "wave-making distance"; the cosine term in the formula is then prefixed by a minus sign instead of a positive sign. We return to this point later; we first work out a definite simple illustration in "two-dimensional waves," and then build up a more complete formula for comparison with experiment. With the same notation as in § 1, let the pressure system be given by

$$p = f(x) = \frac{1}{\pi}\left\{\frac{\mathrm{P}_1\alpha^2}{\alpha^2+(x-\frac{1}{2}l)^2}-\frac{\mathrm{P}_2\alpha^2}{\alpha^2+(x+\frac{1}{2}l)^2}\right\}. \tag{15}$$

* W. Froude, *loc. cit. ante*, p. 83.

This indicates two pressure systems, one of excess and the other of defect of pressure; each distribution is of the type already used, and their centres are separated by a distance l. Fig. 3 shows the character of the disturbance.

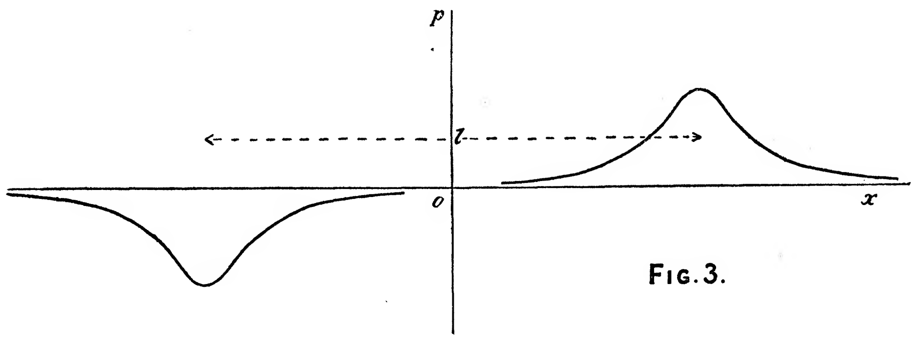


FIG. 3.

In the rear of the whole disturbance there is interference between the regular wave-trains due to the two parts. With the same methods as before we find that the resulting waves are given by

$$\eta = \frac{2gP_1}{wv^2} e^{-ag/v^2} \sin \frac{g(x-\frac{1}{2}l)}{v^2} - \frac{2gP_2}{wv^2} e^{-ag/v^2} \sin \frac{g(x+\frac{1}{2}l)}{v^2}$$

$$= \frac{2g}{wv^2} e^{-ag/v^2} \left\{ (P_1 - P_2) \cos \frac{gl}{2v^2} \sin \frac{gx}{v^2} - (P_1+P_2) \sin \frac{gl}{2v^2} \cos \frac{gx}{v^2} \right\}. \quad (16)$$

Hence the average energy per unit area is proportional to

$$v^{-4} e^{-2ag/v^2} \{P_1^2 + P_2^2 - 2P_1P_2 \cos(gl/v^2)\}.$$

Now, assuming as before that P_1 and P_2 vary as v^2, we find that as regards variation with the velocity the effective resistance R, which is the expression of the energy required to feed the wave-trains, is given in the form

$$R = \{A^2 + B^2 - 2AB \cos(gl/v^2)\} e^{-2ag/v^2}. \quad (17)$$

A more general expression might have been obtained by taking two quantities α_1 and α_2 in (15), corresponding to some difference in wave-making properties of entrance and run; this would have led to different exponential factors being attached to the bow and stern waves. However, we find (17), with a common exponential factor, sufficiently adjustable for present purposes.

In Froude's experiments in 1877 the effect of inserting different lengths of parallel middle body between the same entrance and run was examined; it was found that a hump in the residuary resistance curve corresponded to a trough of the bow waves being in the vicinity of the middle of the run, and a hollow to a crest being in that position.

For the model, Ship A, we have: Length = L = 400 feet; entrance = run = 80 feet.

Hence, in this case we may take, in formula (17), l as approximately 360 feet. We notice that this gives $l = 0{\cdot}9\mathrm{L}$; and in subsequent comparisons, instead of leaving l to be adjusted to fit the experimental curve, we find there is sufficient agreement if we fix it beforehand as 0·9 of the length of the ship on the water-line.

Compare, now, the length l with the ordinary "wave-making length" of the ship; the latter is written as $m\mathrm{L}$ and is defined as the distance between the first regular bow crest and the first regular stern crest. From the present point of view (17) gives

$$m\mathrm{L} = l + \tfrac{1}{2}\lambda \quad \text{or} \quad m = 0{\cdot}9 + \tfrac{1}{2}\lambda/\mathrm{L}, \tag{18}$$

where λ is the wave-length in feet of deep-sea waves of velocity v ft./sec.

Calculating from this formula for Ship A, and writing V for velocity in knots (6080 feet per hour), we obtain Table III.

We see that the statement that m lies between 1 and about 1·2 would hold for this ship if it were measured for ordinary speeds between about 14 and 22 knots.

Table III.

V.	λ.	m.
10	55·5	0·97
14	110	1·03
18	180	1·12
22	270	1·24
26	362	1·35
30	500	1·5

We proceed now to modify (14) by introducing into the second term a factor $1 - \gamma \cos(gl/v^2)$. With $l = 360$, we find gl/v^2 is approximately $4080/\mathrm{V}^2$, with V in knots; further, from one value from the experimental curve we obtain $\gamma = 0{\cdot}12$. Thus for Ship A we have R in tons given by

$$\mathrm{R} = 4{\cdot}5e^{-\frac{2}{3}(26/3\mathrm{V})^2} + 297\,\{1 - 0{\cdot}12\cos(4080/\mathrm{V}^2)\}\,e^{-\frac{2}{3}(26/\mathrm{V})^2}. \tag{19}$$

Table IV shows some calculated values for R, and these are represented in fig. 4 by dots; the continuous curve is the experimental residuary resistance curve given by Froude, that is, the total resistance less the calculated frictional part.

It is the custom to give the results of model experiments in the form of a fair curve, so that the positions of actual readings and the possible

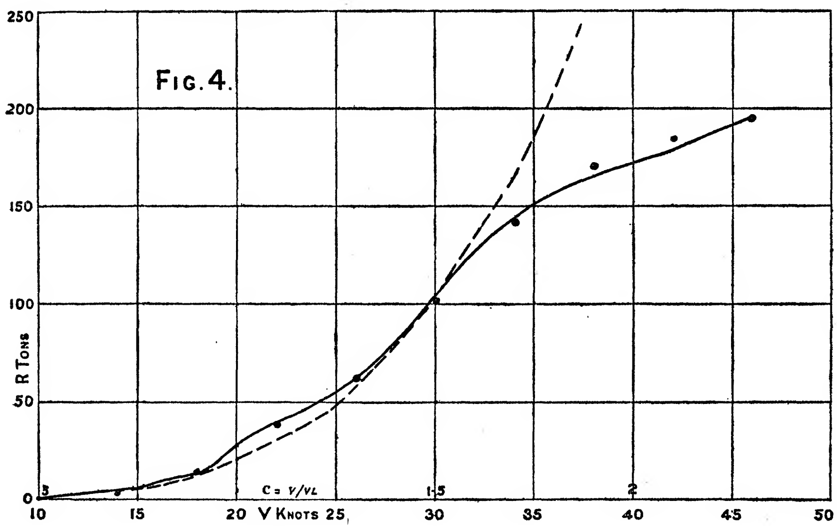


Fig. 4.

error are not known. The interrupted curve is a curve $R = AV^4$ sketched in for comparison.

Table IV.

V.	R.	V.	R.
10	1·5	30	102
14	4·2	34	142
18	15	38	171
22	44	42	185
26	62	46	195

§ 5. *Comparison with Experimental Results.*

Before examining further model curves we must express the previous formula in a form more suitable for calculation; we use the system of units in which model results are now generally expressed. R is given in lbs. per ton displacement of the ship, while instead of the speed V we use the ratio $V/\sqrt{L}$, V being in knots and L in feet; this is called the speed-length ratio, and we shall denote it by c. The advantage of these units is that they utilise Froude's law of comparison; from the experimental curve between R and c we can write down at once the residuary resistance for a ship of any length and displacement at the corresponding velocity, provided the ship has the same lines and form as the model. Thus the constants which are left in the relation between R and c depend only upon the lines of the model, not upon its absolute size. At present we make no attempt to connect these constants with the form of the model, as expressed by the usual coefficients

of fineness or the curve of sectional areas, or in other ways; we are concerned with the form of R as a function of c, and the constants are chosen in each case to make the best fit possible.

First, as regards the exponential factor, we had $e^{-\frac{3}{2}(\mathrm{V}'/\mathrm{V})^2}$, with V′ giving a point of inflection on the resistance curve; in the case of Ship A we had $\mathrm{V}' = 26$, $\mathrm{L} = 400$, so that $c' = 1{\cdot}3$. Now, it is just about this value of c that there is a falling off in most experimental curves, so that we try first $c' = 1{\cdot}3$ for the point of inflection on the R, c curve. Then the exponential factor becomes $e^{-\frac{3}{2}(c'/c)^2}$, or $e^{-2{\cdot}53/c^2}$.

Secondly, as regards the cosine term which gives the undulations, we had $\cos(gl/v^2)$; we have decided to put $l = 0{\cdot}9\mathrm{L}$, so that we have

$$\frac{gl}{v^2} = 0{\cdot}9g\mathrm{L}\bigg/\left(\frac{6080}{3600}\mathrm{V}\right)^2 = \frac{10{\cdot}2}{c^2}, \text{ approximately.}$$

Hence the previous relation for R reduces to the following general form:

$$\mathrm{R} = \alpha e^{-2{\cdot}53/9c^2} + \beta(1-\gamma\cos 10{\cdot}2/c^2)\, e^{-2{\cdot}53/c^2}, \tag{20}$$

where R is in lbs. per ton displacement, and α, β, γ depend upon the form of the model.

There are humps on the curve when $10{\cdot}2c^{-2}$ is an odd multiple of π, hollows when it is an even multiple, and mean values when it is an odd multiple of $\frac{1}{2}\pi$. For facilitating calculation, some of these positions are given in Table V; and, for the same reason, values of the exponentials and the cosine factor are given in Table VI.

Table V.

Humps	—	—	1·8	—	—	—	1·04	—	—	—	0·8		
Means ...	—	2·54	—	1·47	—	1·13	—	0·96	—	0·85	—	0·76	
Hollows	∞	—	—	—	1·27	—	—	—	0·9	—	—	—	0·73

Values of c.

Table VI.

c.	$e^{-2{\cdot}53/9c^2}$.	$e^{-2{\cdot}53/c^2}$.	$\cos(10{\cdot}2/c^2)$.
0·6	0·460	0·0009	+0·75
0·8	0·644	0·019	−0·97
1·0	0·756	0·080	−0·71
1·2	0·821	0·172	+0·70
1·4	0·866	0·275	+0·47
1·6	0·896	0·372	−0·65
1·8	0·916	0·458	−1·0
2·0	0·932	0·532	−0·83
2·2	0·943	0·592	−0·51
2·4	0·951	0·644	−0·20
3	0·970	0·756	+0·43

We examine, now, some examples of experimental curves, comparing them with the formula (20); several of the curves and other data, in particular for II, III, and V, have been taken from the collection in Prof. Hovgaard's paper already referred to, in which he essays to fit formulæ involving V^4 or V^6 with the experimental curves.

I. *R. E. Froude*, 1881, *Ship A*.

Displacement = 4090 tons; length = 400 feet; cylindrical coefficient = 0·694.

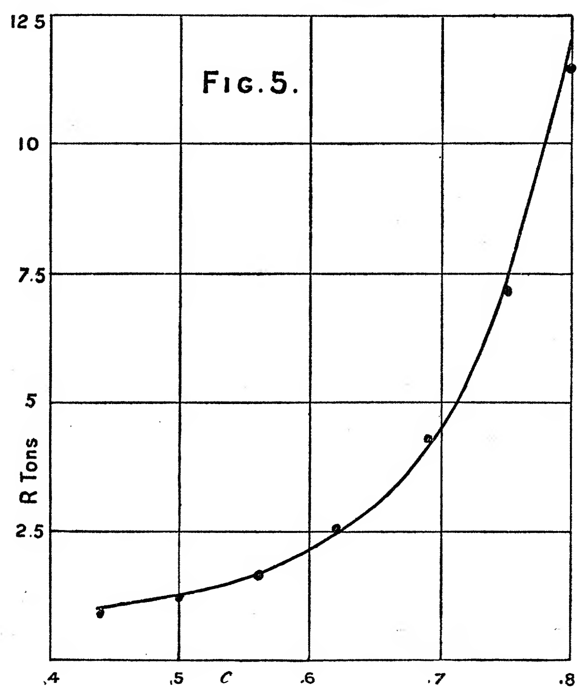
This is the case we have examined in the previous sections, so that we have only to change the numerical factors in (19) to cause R to be given in lbs. per ton displacement. We find the result is formula (20) with

$$\alpha = 2{\cdot}46; \quad \beta = 162{\cdot}6; \quad \gamma = 0{\cdot}12.$$

II. *W. Froude*, 1877.

Displacement = 3804 tons; length = 340 feet; cylindrical coefficient = 0·787.

The last two data include the cylindrical middle body. The curve is given in fig. 5; it was constructed by Hovgaard from the data of Froude's



experiments, and these were such that it was possible to make a mean residuary resistance curve, the effects of bow and stern interference being eliminated. The curve is given as total residuary resistance in tons on a base of V in knots. If we work in lbs. per ton, we find there is a very fair agreement with formula (20) if we take

$$\alpha = 2{\cdot}24; \quad \beta = 279{\cdot}7; \quad \gamma = 0.$$

Probably a closer agreement could be obtained by further slight adjustment of α and β. Fig. 5 shows a comparison of values of the total residuary resistance for the ship (in tons); the calculated values are indicated by small circles.

III. *D. W. Taylor*, 1000 *lbs. Model.*

Length on water line = 20·51 feet; cyl. coeff. = 0·680.

The experimental curve in this case is given as residuary resistance for the model in lbs. on a base of V in knots. With the same notation as before we find

$$\alpha = 2; \quad \beta = 136{\cdot}6; \quad \gamma = 0{\cdot}14.$$

Putting these values in (20), we can calculate R in lbs. per ton, and hence R_1 in lbs. for the model; fig. 6 shows the comparison between R_1 and the corresponding values on the curve; the calculated values R_1 are indicated by dots.

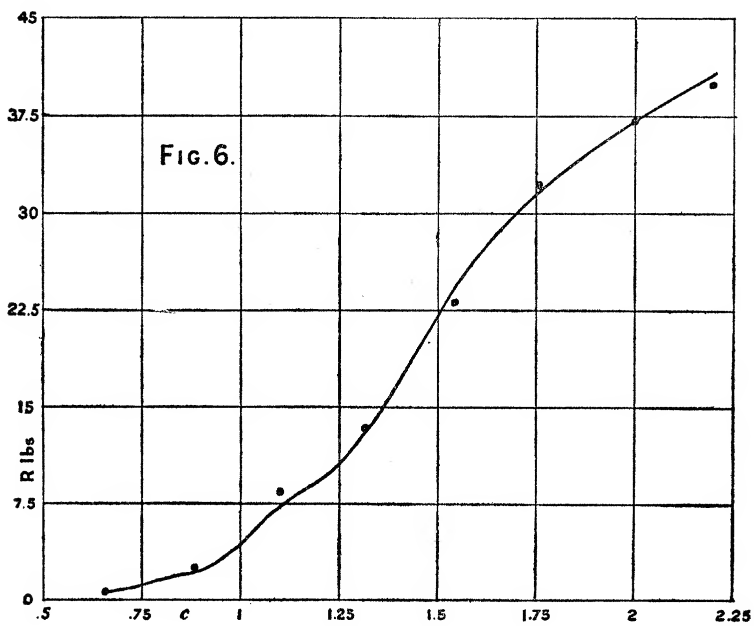


Fig. 6.

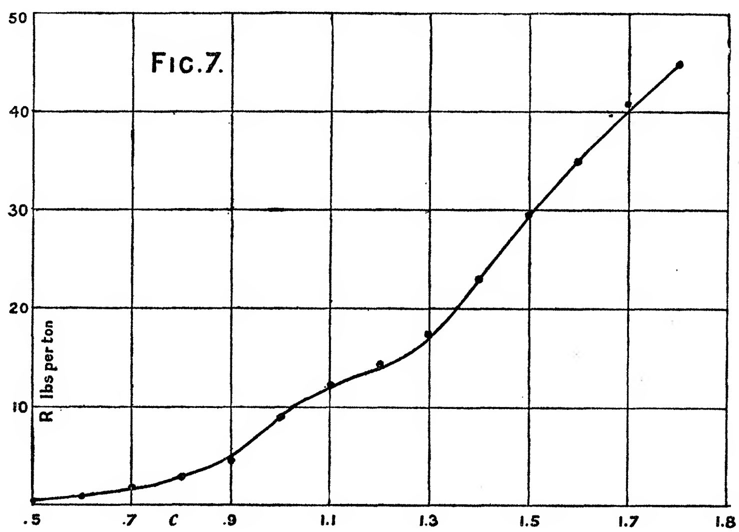
IV. *D. W. Taylor, Model No.* 892.*

Displacement = 500 lbs.; length on water line = 20·512 feet; longitudinal coeff. = 0·68; midship section coeff. = 0·70.

In this case the experimental curve is given as lbs. per ton displacement (R′) on a base of speed-length ratio (c). In the same manner as before, fig. 7 shows the comparison with the formula (20) when we take

$$\alpha = 2; \quad \beta = 82{\cdot}5; \quad \gamma = 0{\cdot}14.$$

Since the constant α is small compared with β, one is not able to lay much stress on the meaning of the first term. For as the velocity functions



are of a suitable type, the constants possess considerable elasticity as regards fitting an experimental curve. For instance, if we omit values of c below about 0·9, it is possible to represent the previous curves fairly well by a formula

$$\mathrm{R} = \beta\,\{1 - \gamma\cos(10{\cdot}2/c^2)\}\,e^{-\frac{3}{2}(c'/c)^2}.$$

In the previous examples we took the value 1·3 for c'. In Case IV above we find now the values

$$\beta = 87; \quad \gamma = 0{\cdot}14; \quad c' = 1{\cdot}3.$$

For a similar curve taken from the same paper, viz., Model No. 891, displacement 1000 lbs., we find a good correspondence, except for slightly higher values near $c = 1{\cdot}1$, with the values

$$\beta = 174; \quad \gamma = 0{\cdot}14; \quad c' = 1{\cdot}4.$$

* D. W. Taylor, Society of Naval Architects, New York, November 19, 1908.

V. *I. I. Yates, Destroyer Model C.**

Displacement = 575 lbs.; length = 20 feet; cyl. coeff. = 0·529.

The experimental curve is given in lbs. for the model on a base of V in knots, and is a total resistance curve, that is, it includes the frictional resistance. The curve is reproduced in fig. 8.

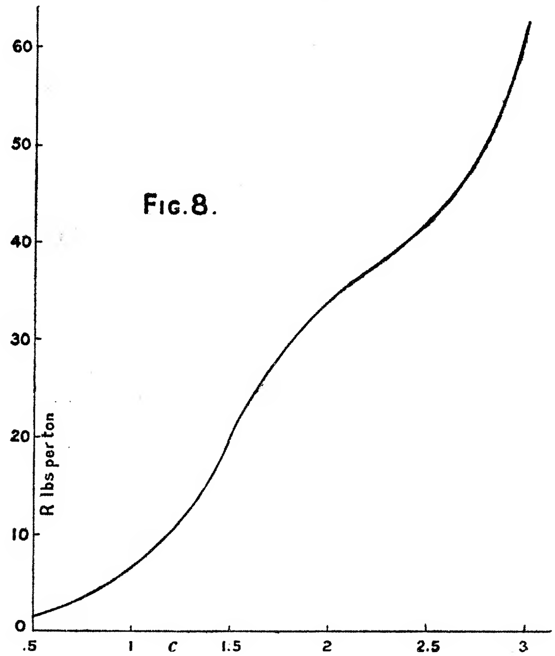


Fig. 8.

This curve is not analysed here so as to compare the residuary resistance with the formula (20), but it is included in order to draw attention to certain possible complications. It may be noticed that the curve is carried to a high value of the speed-length ratio c, and that it continues to rise more rapidly after about $c = 2{\cdot}3$ than might be expected on the present theory. Now in the first place it is possible that the frictional resistance may account partly for this rise. The ordinary estimation of the frictional resistance assumes that it can be calculated separately from some expression like $f\,\mathrm{SV}^{1{\cdot}85}$; now the legitimacy of this is beyond doubt in all ordinary cases, but at high speeds it is possible that the form of the expression may change, or even that it may not be a fair simplification to divide the total resistance into simple additive components.

In the second place a more important consideration must be taken into account, and that is the depth of the tank. For the experiments now under

* I. I. Yates, Thesis, 1907, Mass. Inst. Tech. U.S.A. See Hovgaard, *loc. cit. ante.*

consideration the depth of water in the tank is not known. The deepest experimental tank appears to be the U.S. Government tank at Washington, which has a maximum depth of about 14·7 feet. Now in that tank, with a 20-foot model, there would be a "critical" condition near the value $c = 2{\cdot}9$; before and up to that point the residuary resistance curve would rise sharply and abnormally. This effect is discussed more fully in the next section, and curves are given in fig. 11, with which fig. 8 may be compared. It appears, then, as far as one is able to judge, that it is possible the resistance curve in fig. 8 is complicated by the effect of finite depth of the tank.

§ 6. *The Effect of Shallow Water.*

We saw in the first section that the wave-making resistance R can be written in the form

$$\mathrm{R} = \tfrac{1}{2} w a^2 (v-u)/v,$$

where u is the group-velocity corresponding to wave-velocity v. For deep water $u = \frac{1}{2}v$, and the formulæ are comparatively simple. But for water of finite depth h the relation between u and v depends upon the wave-length $(2\pi/\kappa)$. We have

$$v = \sqrt{\left(\frac{g}{\kappa}\tanh \kappa h\right)},$$

$$u = \frac{d}{d\kappa}(\kappa v) = \tfrac{1}{2}v\,(1+2\kappa h/\sinh 2\kappa h).$$

Consequently we find

$$\mathrm{R} = \tfrac{1}{4} w a^2 \left(1-\frac{2\kappa h}{\sinh 2\kappa h}\right). \tag{21}$$

As v increases from zero to $\sqrt{(gh)}$, R diminishes from $\frac{1}{4}wa^2$ to 0, provided the amplitude remains constant. But as Prof. Lamb remarks,* the amplitude due to a disturbance of given character will also vary with the velocity. It is the variation of this factor that we have to examine in the manner used in the previous sections for deep water.

If a symmetrical line-pressure system $\mathrm{F}(x)$, suitable for Fourier analysis, is moving uniformly with velocity v over the surface of water, the surface disturbance η is given by

$$\pi w \eta = \tfrac{1}{2}\int_0^\infty dt \int_0^\infty \kappa \mathrm{V} \phi(\kappa) \sin \kappa\,\{x+(v-\mathrm{V})\,t\}\,d\kappa$$

$$-\tfrac{1}{2}\int_0^\infty dt \int_0^\infty \kappa \mathrm{V} \phi(\kappa) \sin \kappa\,\{x+(v+\mathrm{V})\,t\}\,d\kappa, \tag{22}$$

where $\phi(\kappa) = \int_{-\infty}^{\infty} \mathrm{F}(\omega) \cos \kappa\omega\, d\omega$.

* H. Lamb, 'Hydrodynamics,' p. 391, 1906.

The method of evaluating these integrals approximately so as to give the regular wave-trains has been discussed in a previous paper and it is followed now in the case of finite depth.* We take, under certain limitations, the value of an integral such as

$$y = \int \phi(u) \sin\{g(u)\}\, du$$

to be the value of its principal group, viz.,

$$y_0 = \left\{\frac{2\pi}{g''(u_0)}\right\}^{\frac{1}{2}} \phi(u_0) \cos\{g(u_0) - \tfrac{1}{4}\pi\}, \tag{22A}$$

where u_0 is such that $g'(u_0) = 0$.

Now in the integrals in (22) we have to find successively two principal groups, first with regard to κ and then in the variable t; and thus we may evaluate the amplitude factor in the resulting regular wave-trains.

For water of depth h we may write

$$f(\kappa) = v - \mathrm{V} = v - \sqrt{\left(\frac{g}{\kappa} \tanh \kappa h\right)}.$$

The group with respect to κ gives a term proportional to

$$\cos\{t\kappa^2 f'(\kappa) + \tfrac{1}{4}\pi\},$$

where κ has the value given by

$$f(\kappa) + \kappa f'(\kappa) = -\frac{x}{t}. \tag{23}$$

From (22A), this introduces into the amplitude a factor

$$1/\sqrt{[t\{2f'(\kappa) + \kappa f''(\kappa)\}]}. \tag{24}$$

Further, the group with respect to t occurs for

$$\frac{d}{dt}\{t\kappa^2 f'(\kappa)\} = 0 \quad \text{or} \quad f(\kappa) = 0.$$

Also we have in these circumstances

$$\frac{d^2}{dt^2}\{t\kappa^2 f'(\kappa)\} = \frac{d}{dt}\left\{\kappa^2 f'(\kappa) + \frac{\kappa x}{t}\right\} = \frac{d}{dt}\{-\kappa f(\kappa)\}$$

$$= -\frac{x}{t^2}\frac{f+\kappa f'}{2f'+\kappa f''} = \frac{1}{t}\frac{(f+\kappa f')^2}{2f'+\kappa f''} = \frac{(\kappa f')^2}{t(2f'+\kappa f'')}. \tag{25}$$

Hence from (22A), (24), and (25) the selection of the two groups adds to the amplitude a factor $1/\kappa f'(\kappa)$, where

$$f(\kappa) = 0 = v - \sqrt{\left(\frac{g}{\kappa} \tanh \kappa h\right)}.$$

* Havelock, 'Roy. Soc. Proc.,' A, vol. 81, p. 411, 1908.

Also if u is the group-velocity for wave-length $2\pi/\kappa$ and wave-velocity V, we have, in this case,

$$u = \frac{d}{d\kappa}(\kappa \mathrm{V}) = \frac{d}{d\kappa}\{\kappa v - \kappa f(\kappa)\} = v - \{f(\kappa) + \kappa f'(\kappa)\}.$$

Hence, since in the final value $f(\kappa) = 0$, we have $\kappa f'(\kappa)$ equal to $v-u$. Thus if κ is the wave-length of the regular wave-trains in the rear of the disturbance, we find that they are given by

$$\eta = \text{const.} \times \frac{\kappa v \phi(\kappa)}{v-u} \sin \kappa x, \tag{26}$$

where $$v = \sqrt{\left(\frac{g}{\kappa} \tanh \kappa h\right)}, \qquad u = \tfrac{1}{2} v \left(1 + \frac{2\kappa h}{\sinh 2\kappa h}\right).$$

Hence for the amplitude a we have

$$a = \mathrm{C}\kappa \phi(\kappa) \bigg/ \left(1 - \frac{2\kappa h}{\sinh 2\kappa h}\right).$$

Substituting now in (21) we obtain for the wave-making resistance, R proportional to

$$\kappa^2 \{\phi(\kappa)\}^2 \bigg/ \left(1 - \frac{2\kappa h}{\sinh 2\kappa h}\right).$$

If we take the same distribution of pressure in the travelling disturbance, namely, $\mathrm{F}(x) = \mathrm{P}\alpha/\pi(\alpha^2 + x^2)$, we have $\phi(\kappa) = \mathrm{P}e^{-\alpha\kappa}$; further, we may again assume that the pressure P varies as v^2, so that we have the resistance in the form

$$\mathrm{R} = \mathrm{A}\, \kappa^2 v^4 e^{-\beta\kappa} \bigg/ \left(1 - \frac{2\kappa h}{\sinh 2\kappa h}\right),$$

with $$\frac{\tanh \kappa h}{\kappa h} = \frac{v^2}{gh}. \tag{27}$$

Considering R given as a function of v by these two equations, we see that R increases slowly at first and then rapidly up to a limiting value at the critical velocity $\sqrt{(gh)}$; after this point R is zero, for there is no value of κ satisfying the second equation with $v^2/gh > 1$.

Further, the limiting value of R at the critical velocity is finite, for we have

$$\lim_{\kappa=0} \frac{\kappa^2 h^2}{(1 - 2\kappa h/\sinh 2\kappa h)} = 1{\cdot}5.$$

We see that the R, v curve given by (27) is of the type sketched in fig. 9. We may compare this with some of the curves given by Scott Russell for canal boats. The continuous curve in fig. 10 is an experimental curve of

total resistance,* and the dotted curve is a parabolic curve inserted here to represent approximately the frictional resistance; the difference between the two curves represents the residuary resistance, and is clearly of the same type as the theoretical curve in fig. 9.

FIG. 9.

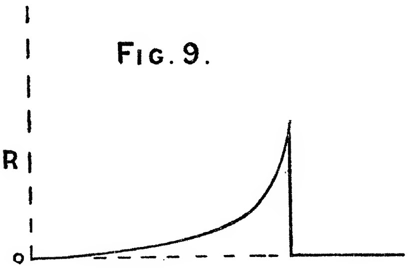
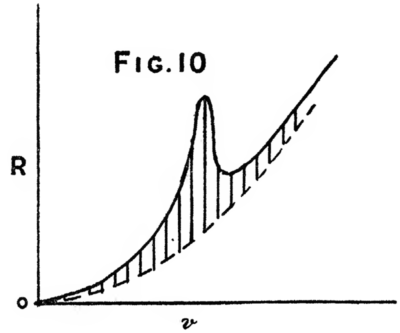


FIG. 10



We can obtain a better estimate of equation (27) by taking an experimental curve for a model in deep water, and then building up curves for different depths. We must first put (27) into a form suitable for comparison with deep water results.

Limiting the problem to one of transverse waves only, the formula (27) must reduce to $R = Ae^{-2 \cdot 53/c^2}$, for h infinite and $c =$ (speed in knots)/$\sqrt{}$(length in feet).

Writing v' for $v/\sqrt{(gh)}$ we find $c^2 = 11 \cdot 3v'^2h/L$; thus although the actual critical velocity does not depend upon the length of the ship but only on the depth of water, the speed-length ratio (c) has a critical value which is proportional to the square root of the ratio (depth of water)/(length of ship).

In (27) we cannot fix any value of v or c and then calculate R directly; we must work through the intermediate variable κh. The equations may now be written as

$$R = A(\kappa h)^2 v'^4 e^{-\beta\kappa h}/(1-2\kappa h/\sinh 2\kappa h), \qquad (28)$$

$$v'^2 = (\tanh \kappa h)/\kappa h; \qquad \beta' = 0 \cdot 218L/h; \qquad c^2 = 11 \cdot 3v'^2h/L.$$

With h infinite this reduces to the previous form for deep water with the same constant A, so that a direct comparison is possible. As the velocity v increases from 0 to $\sqrt{(gh)}$, κ diminishes from ∞ to 0; we select certain values of κh, calculate the values from tables of hyperbolic functions, and thus obtain the set of values in Table VII, writing m for

$$(\kappa h)^2 v'^4/(1-2\kappa h/\sinh 2\kappa h).$$

* J. Scott Russell, 'Edin. Phil. Trans.,' vol. 14, p. 48, 1840.

Table VII.

κh.	$v/\sqrt{(gh)}$.	$c^2\mathrm{L}/h$.	m.	$-\beta\kappa c^2$.
∞	0	0	1·0	2·53
10	0·316	1·13	1·0	2·53
6	0·41	1·87	1·0	2·53
4	0·5	2·82	1·005	2·53
2	0·69	5·42	1·077	2·43
1	0·87	8·57	1·287	1·92
0	1·0	11·3	1·5	0

We consider now the experimental curve analysed in Case IV in the previous section, a model of 20·5 feet taken up to a value $c = 1·8$. Assuming that the influence of finite depth was inappreciable in this range, we have for deep water

$$\mathrm{R} = 2e^{-2·53/9c^2} + 82·5\,\{1 - 0·14\cos(10·2/c^2)\}\,e^{-2·53/c^2}. \tag{29}$$

We leave out of consideration at present the first term, which is supposed to represent the diverging waves, and we extend the calculations for R (transverse) from the rest of the formula up to $\mathrm{C} = 3·3$ taken at intervals of 0·1 for C; we obtain thus the lowest curve given in fig. 11. With the help of Table VII, we calculate values of R for depths of about 5, 10, 12, 15, and 20 feet, taking in the formula (28) A equal to

$$82·5\,\{1 - 0·14\cos(10·2/c^2)\}$$

so that the results apply to the same model at different depths. An example of the calculations for one case may be sufficient; Table VIII shows the intermediate steps for $h = 12·3$ ft., $\mathrm{L} = 20·5$.

Table VIII.

c^2.	c.	$-\beta\kappa$.	R/A.	$e^{-2·53/c^2}$.
0·68	0·825	3·73	0·024	0·024
1·12	1·06	2·26	0·106	0·106
1·69	1·3	1·5	0·224	0·223
3·25	1·8	0·75	0·508	0·472
5·14	2·27	0·374	0·385	0·687
6·8	2·61	0	1·5	1

The results for the five values of h are given in Table IX, and from these the curves in fig. 11 have been drawn.

The general character of the effect of finite depth is clear on inspection of the set of curves in fig. 11. If it is required to go to high values of the speed-length ratio in a given tank, the ratio of the depth of water to the length of the model must be adjusted so that there is no appreciable effect in

Table IX.

h/L.							
1	c R	0·7 0·7	1·0 7·5	1·5 27·5	1·7 39·2	2·33 59·2	3 80·5
0·75	c R	0·7 0·7	1·0 7·5	1·2 13	1·4 21·4	2 54·5	2·54 79·2
0·6	c R	0·7 0·7	1·0 7·5	1·3 17·9	1·8 47·7	2·3 78	2·6 122
0·5	c R	0·7 0·7	1·0 7·5	1·2 13·1	1·65 40	2·1 74	2·38 127
0·25	c R	0·7 0·7	0·8 1·7	0·84 3·3	1·16 14·1	1·46 43·6	1·68 142·5

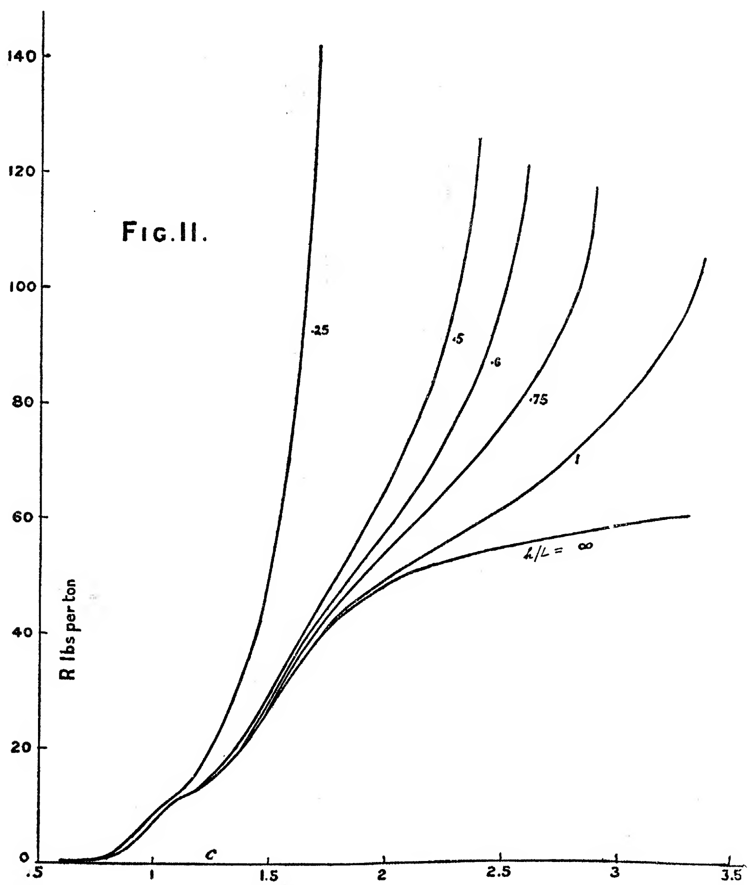


Fig. II.

the range of the experiments. Since the curves given here are theoretical curves for transverse waves only, each of them ends abruptly at the critical

velocity—the resistance being zero after that point. In practice, we know that there are no such discontinuities in the resistance curves, and there are certain considerations which go to account for this difference. First, as regards the transverse waves alone, the preceding formulæ show that the amplitude tends to become infinite at the critical velocity, although the corresponding resistance at uniform velocity remains finite; but, even apart from the effects of viscosity, there is a highest possible wave with a velocity depending partly upon the amplitude. Secondly, we have left out of consideration the diverging waves; but these must become more important in the neighbourhood of the critical velocity, for we may regard the two systems as coalescing into one solitary wave in the limit as the critical velocity is reached. After this point the diverging waves persist, so that the effect of these would be of the order of halving the drop in the resistance as the critical velocity is passed.

Finally, we must consider the frictional resistance, which increases steadily with the velocity; so that the fall is finally a smaller percentage of the total resistance than might appear at first. The curves given in fig. 11 give an estimate of a maximum effect of this kind, considering only the transverse wave system.

§7. *Further Types of Pressure Distribution.*

The preceding formulæ have been built up on the effect of a travelling pressure disturbance of simple type; we consider now another type which we may use as an illustration.

Let the pressure system be given by

$$p = f(x) = \mathrm{A}\,(h^2-x^2)/(x^2+h^2)^2.$$

The type of distribution is graphed in fig. 12.

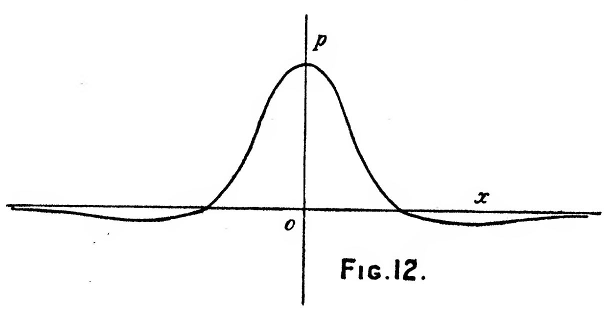


FIG. 12.

Proceeding as in §2, we have

$$\phi(\kappa) = 2\mathrm{A}\int_0^\infty \frac{h^2-\omega^2}{(\omega^2+h^2)^2}\cos\kappa\omega\, d\omega = \pi \mathrm{A}\kappa e^{-\kappa h}. \tag{30}$$

Hence the amplitude of the regular wave-trains formed on deep water in the rear of this disturbance is proportional to $\kappa^2 A e^{-\kappa h}$, and the effective wave-making resistance is proportional to $\kappa^4 A^2 e^{-2\kappa h}$. We make the same assumption as before, viz., A proportional to v^2, and write $\kappa = g/v^2$; then the resistance is given by

$$R = Cv^{-4} e^{-2gh/v^2}. \tag{31}$$

We use this expression to show how R varies with the constant h of the pressure system. Let $v = 10$ ft./sec., and let $R = 1$ for $h = 0$; then we find the following relative values:

h.	R.
0	1·0
1	0·52
5	0·04
10	0·0016

R decreases very rapidly as h is increased. We have chosen this example for the following reason. Consider the motion of a thin infinite cylinder in an infinite perfect fluid; if we consider a plane parallel to the direction of motion and to the cylinder and at a distance h from it, we find that the distribution of excess or defect of pressure due to the motion is of the above type. Now, this is not the same as a cylinder moving in deep water at a depth h below the free surface, but it is suggested that as a first approximation the wave-forming effect is that of an equivalent diffused pressure system. The illustration shows how rapidly the wave-making resistance diminishes with the amount of diffusion, that is, with the depth h; this, of course, agrees with the experiments on the resistance to motion of submerged bodies, and, in fact, with the resistance of submarine vessels.

In the preceding work no attempt has been made to connect theoretically the constants in the pressure formula with those of the model; since the theory rests chiefly on the consideration of transverse waves only, this would presumably bring into question the length of entrance, run, and so forth. The consideration of any "transverse" constants, such as the beam, would need a fuller treatment of a diffused pressure system in two dimensions on the surface so as to give a more detailed investigation of both transverse and diverging wave systems.